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## Dirac Quasinormal modes of MSW black holes

Saneesh Sebastian\*

*Department of Physics, Cochin University of Science and Technology, Kochi 682022, India*  
*saneeshphys@cusat.ac.in*

V C Kuriakose

*Department of Physics, Cochin University of Science and Technology, Kochi 682022, India*  
*vck@cusat.ac.in*

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In this paper we study the Dirac quasinormal modes of an uncharged  $2+1$  black hole proposed by Mandal et. al and referred to as MSW black hole in this work. The quasinormal mode is studied using WKB approximation method. The study shows that the imaginary part of quasinormal frequencies increases indicating that the oscillations are damping and hence the black hole is stable against Dirac perturbations.

*Keywords:* Black hole, MSW black hole, Quasinormal modes, Dirac field.

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### 1. Introduction

The study of quasinormal modes has a long history<sup>1</sup>. Vishveshwara<sup>2</sup> first presented the idea of quasinormal modes. Chandrasekhar and Detweiler<sup>3</sup> studied the quasinormal modes of Schwarzschild black hole. The study of quasinormal modes started with black hole stability problem which concern the evolutions of black hole perturbations. The evolution of wave fields around a black hole consists of roughly three stages<sup>4,5,6</sup>. The first is an initial wave burst coming directly from source and is dependent on the initial form of the original wave field. The second part involves damped oscillations called quasinormal frequencies, which is independent of the initial value of the wave but depends on the background black hole spacetime. Because of radiation damping, the normal frequencies are complex. The last stage is the power law tail which arises because of back scattering of long range gravitational field<sup>7</sup>.

It is believed that the presence of black hole can be inferred using this quasinormal waves. Thus recently the study of quasinormal modes becomes important

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as a number of gravitational wave detectors are expected to start operating soon. Usually the quasinormal mode frequencies are damped complex frequencies and thus we must employ the numerical methods to evaluate them. Unlike the case of gravitational, electromagnetic and scalar perturbations, the Dirac field is more complicated to evaluate. The third order WKB method<sup>8,9,10</sup> is utilized for computation of quasinormal modes in the present study.

2 + 1 dimensional black holes are the simplest toy model of general 3 + 1 black holes in General Theory of Relativity. One example of 2 + 1 black holes is the BTZ black holes<sup>11</sup>. The scalar, electromagnetic and gravitational perturbations can be analytically studied for such black holes<sup>12</sup>, but it is difficult in the case of Dirac field. Such studies using WKB method are there in the literature. Another type of 2 + 1 black holes is charged dilaton black holes. Quasi normal modes of charged dilaton black hole is studied<sup>13</sup>. A special case of such black holes is the MSW black holes. It is a one parameter family of chargeless black holes proposed by Mandal et. al<sup>14</sup>. Previously we have studied the thermodynamics and spectroscopy of MSW black hole<sup>15</sup>

Here in the present work we study the massless Dirac quasinormal modes of MSW black holes using WKB approximation. The paper is organized as follows in section. 2 we introduce the MSW black hole. In section. 3 we discuss the Dirac field in MSW space-time background. In section. 4 we evaluate the quasinormal modes and finally in section. 5 give the conclusions.

## 2. The MSW black hole

The metric for MSW black hole is given by ( $c = G = 1$  system of units)

$$ds^2 = -U(r)dt^2 + \frac{dr^2}{U(r)} + \gamma^2 r d\phi^2, \quad (1)$$

where  $U(r)$  is given by<sup>16</sup>

$$U(r) = 8\Lambda\beta r - 2M\sqrt{r}. \quad (2)$$

This metric has a zero at (ie the horizon)

$$r_+ = \frac{M^2}{16\Lambda^2\beta^2}, \quad (3)$$

where  $\Lambda$  is the cosmological constant,  $\beta$  is a constant factor,  $M$  is the mass of the black hole.

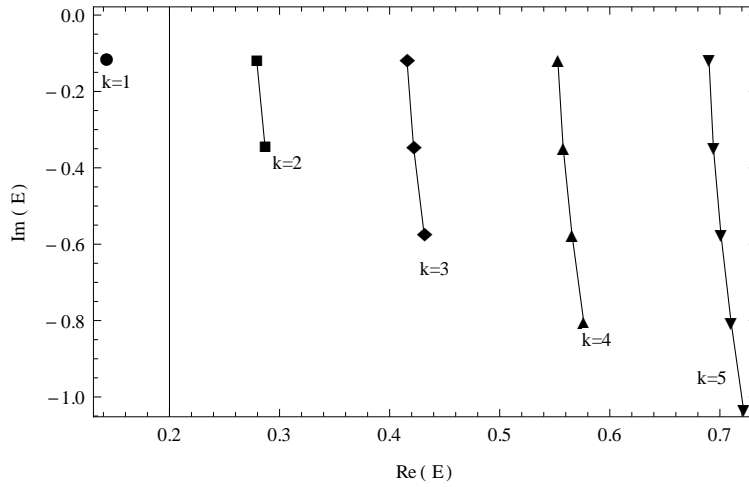
## 3. Dirac field in MSW space-time background

In this section we consider the Dirac field in MSW background space-time. The Dirac equation in a general background space-time can be written as<sup>17</sup>

$$[i\gamma^\mu\partial_\mu - i\gamma^\mu\Gamma_\mu]\Phi = m\Phi, \quad (4)$$

Table 1. Quasinormal frequencies for  $\beta = 0.1$ .

k	n	Re(E)	Im(E)
1	0	0.142987	-0.113155
2	0	0.279143	-0.114674
	1	0.286690	-0.341522
3	0	0.415674	-0.115149
	1	0.421666	-0.343879
	2	0.431245	-0.571293
4	0	0.552724	-0.115311
	1	0.557569	-0.344896
	2	0.565582	-0.573155
	3	0.576227	-0.800933
5	0	0.690033	-0.115378
	1	0.694058	-0.345414
	2	0.700958	-0.574273
	3	0.710081	-0.802461
	4	0.721375	-1.030450

Fig. 1. Quasinormal frequencies of MSW black hole with  $\beta = 0.1$ 

where  $m$  is the mass of the Dirac field,  $\gamma^\mu$  are the curvature dependent Dirac matrices and are represented in terms of tetrad field as

$$\gamma^\mu = e_a^\mu \gamma^a, \quad (5)$$

$\gamma^a$  represent the standard flat space Dirac matrices, which satisfy

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}. \quad (6)$$

$e_a^\mu$ , the tetrad field is defined by

$$g_{\mu\nu} = \eta^{ab} e_\mu^a e_\nu^b, \quad (7)$$

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where  $\eta^{ab} = \text{diag}(-1, 1, 1, 1)$  being the Minkowski metric.  $\Gamma_\mu$ , the spin connections are given by

$$\Gamma_\mu = \frac{1}{8}[\gamma^a, \gamma^b]e_a^\nu e_{b\nu;\mu}, \quad (8)$$

where  $e_{b\nu;\mu} = \partial_\mu e_{b\nu} - \Gamma_{\mu\nu}^\alpha e_{b\alpha}$  is the covariant derivative of  $e_{b\nu}$  with  $\Gamma_{\mu\nu}^\alpha$  being Christoffel symbols. With the MSW metric given as above we can take the tetrad as

$$e_a^\mu = \text{diag}\left(- (8\Lambda\beta r - 2M\sqrt{r})^{-\frac{1}{2}}, (8\Lambda\beta r - 2M\sqrt{r})^{\frac{1}{2}}, 0, r\right) \quad (9)$$

The spin connection satisfy the equation

$$[\Gamma_\mu, \gamma^\nu] = \frac{\partial \gamma^\nu}{\partial x^\mu} + \Gamma_{\mu\rho}^\nu \gamma^\rho. \quad (10)$$

We solve equation (10) for spin connection  $\Gamma_\mu$  and is given by

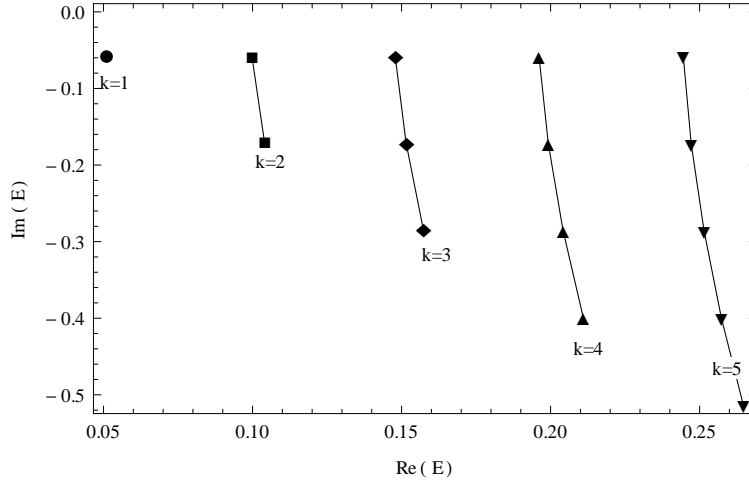
$$\begin{aligned} \Gamma_0 &= -\frac{1}{2} \left( 4\Lambda\beta r - \frac{m}{2\sqrt{r}} \right) (\gamma_1 \gamma_0), \Gamma_1 = 0, \\ \Gamma_2 &= 0, \Gamma_3 = \frac{1}{2} (8\Lambda\beta r - 2m\sqrt{r})^{\frac{1}{2}} (\gamma_1 \gamma_3), \end{aligned} \quad (11)$$

and hence  $\gamma^\mu \Gamma_\mu$  becomes

Table 2. Quasinormal frequencies for  $\beta = 0.05$ .

k	n	Re(E)	Im(E)
1	0	0.051064	-0.056245
2	0	0.099745	-0.056991
	1	0.103979	-0.169241
3	0	0.147834	-0.057383
	1	0.151505	-0.170971
	2	0.157294	-0.284075
4	0	0.196101	-0.057550
	1	0.199216	-0.171805
	2	0.204175	-0.285420
	3	0.210956	-0.398999
5	0	0.244513	-0.057627
	1	0.247177	-0.172257
	2	0.251510	-0.286211
	3	0.257315	-0.399993
	4	0.264698	-0.513824

$$\gamma^\mu \Gamma_\mu = \gamma_1 (8\Lambda\beta r - 2M\sqrt{r})^{\frac{1}{2}} \left[ \frac{1}{2r} + \frac{\left( 4\Lambda\beta - \frac{M}{2\sqrt{r}} \right)}{2(8\Lambda\beta r - 2M\sqrt{r})} \right]. \quad (12)$$

Fig. 2. Quasinormal frequencies of MSW black hole with  $\beta = 0.05$ Table 3. Quasinormal frequencies for  $\beta = 0.01$ .

k	n	Re(E)	Im(E)
3	0	0.0135923	-0.0113035
	1	0.0144644	-0.0334106
	2	0.0158395	-0.0555517
4	0	0.0179184	-0.0113719
	1	0.0187814	-0.0337322
	2	0.0201505	-0.0560795
	3	0.0221412	-0.0784838
5	0	0.0222176	-0.0114231
	1	0.0230410	-0.0339561
	2	0.0243446	-0.0564386
	3	0.0262203	-0.0789680
	4	0.0286961	-0.1015530

Therefor the Dirac equation becomes

$$\left[ i\gamma_0 (8\Lambda\beta r - 2M\sqrt{r})^{\frac{-1}{2}} \partial_t + i\gamma_1 (8\Lambda\beta r - 2M\sqrt{r})^{\frac{1}{2}} \left[ \partial_r + \frac{1}{2r} \right] + i\frac{\gamma_3}{r} \partial_\phi - m \right] \Phi = 0. \quad (13)$$

This equation can be transformed using tortoise coordinates to Schrodinger like equations as,

$$\left( -\frac{d^2}{dr_*^2} + V_{(+1)} \right) F^+ = E^2 F^+ \quad (14)$$

and

$$\left( -\frac{d^2}{dr_*^2} + V_{(+2)} \right) G^+ = E^2 G^+, \quad (15)$$

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where  $r_*$  is the tortoise coordinate,  $F$  and  $G$  are the two wave functions of the Dirac equation.  $V_{(+),1,2}$  are given by

$$V_{(+),1,2} = \pm \frac{dW_{(+)}}{dr_*} + W_{(+)}^2, \quad (16)$$

where  $W$  is a function of  $k$  related to the potential. The same set of equations repeat with  $(+)$  changed to  $(-)$ . Putting together we get

$$\left(-\frac{d^2}{dr_*^2} + V_1\right) F = E^2 F, \quad (17)$$

$$\left(-\frac{d^2}{dr_*^2} + V_2\right) G = E^2 G. \quad (18)$$

The effective potential can be derived from this as

$$V(r, k) = \left(\frac{\sqrt{\Delta}k}{r^2}\right)^2 + \frac{\Delta}{r^2} \frac{d}{dr} \left(\frac{\sqrt{\Delta}k}{r^2}\right). \quad (19)$$

For our metric given by Eq.(1&2) the potential can be written as

$$V(r, k) = \frac{k^2}{r^2} (8\Lambda\beta r - 2M\sqrt{r}) + \frac{k}{2r^2} (8\Lambda\beta r - 2M\sqrt{r})^{\frac{1}{2}} (-8\Lambda\beta r + 3M\sqrt{r}). \quad (20)$$

Fig.(3) shows the dependence of potential on the angular momentum quantum

The potential is plotted for different values of  $k$  is given in figure

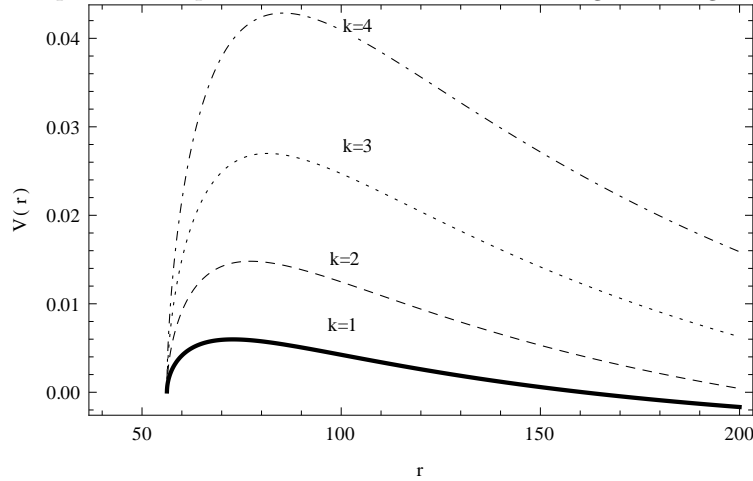


Fig. 3. Variation of potential with  $k$

number  $k$ . It is of barrier form and the barrier increases in height as the  $k$  value increases.

#### 4. Evaluation of Quasinormal Modes for massless Dirac Field

In order to evaluate the Dirac quasinormal modes we use the WKB approximation method developed by Schutz, Will and Iyer<sup>8,9,10</sup>. Comparing with other numerical methods this method has been found to be accurate up to around one percent for both real and imaginary parts. The equation for computing the quasinormal modes  $E$  is given by,

$$E^2 = [V_0 + (-2V_0'')^{\frac{1}{2}}\Lambda] - i(n + \frac{1}{2})(-2V_0'')^{\frac{1}{2}}(1 + \Omega), \quad (21)$$

where

$$\Lambda = \frac{1}{(-2V_0'')^{\frac{1}{2}}} \left[ \frac{1}{8} \left( \frac{V_0^{(4)}}{V_0''} \right) \left( \frac{1}{4} + \alpha^2 \right) - \frac{1}{288} \left( \frac{V_0'''}{V_0''} \right)^2 (7 + 60\alpha^2) \right], \quad (22)$$

and

$$\begin{aligned} \Omega = \frac{1}{(-2V_0'')} & \left[ \frac{5}{6912} \left( \frac{V_0'''}{V_0''} \right)^4 (77 + 188\alpha^2) - \frac{1}{384} \left( \frac{(V_0''')^2 (V_0^{(4)})}{(V_0'')^3} \right) (51 + 100\alpha^2) \right. \\ & + \frac{1}{2304} \left( \frac{V_0^{(4)}}{V_0''} \right)^2 (67 + 68\alpha^2) + \frac{1}{288} \left( \frac{V_0''' V_0^{(5)}}{(V_0'')^2} \right) (19 + 28\alpha^2) \\ & \left. + \frac{1}{288} \left( \frac{V_0^{(6)}}{V_0''} \right) (5 + 4\alpha^2) \right], \end{aligned}$$

where  $\alpha = n + \frac{1}{2}$ , and

$$V_0^{(n)} = \frac{d^n V}{dr^n} \Big|_{r_* = r_{*max}}. \quad (24)$$

Substituting the potential  $V(r)$  obtained from (Eq.20) in Eq.(21) we get the complex quasinormal modes for massless Dirac field in MSW space-time. The values are listed in tables and plotted. Fig.(1) and Fig.(2) represent the quasinormal mode frequencies for different values of  $\beta$ . The figures show that the real part of complex frequencies slightly increase, while the magnitude of imaginary part of frequencies increases rapidly with mode number  $n$  for the same value of  $k$ . This indicates that the higher modes decay faster than the low lying modes.

#### 5. Conclusion

In this work we have studied the quasinormal modes of a chargeless  $2 + 1$  black hole called MSW black hole. The complex frequencies are found out and tabulated. Graphs are plotted for various values of the parameter  $\beta$ . The modulus value of imaginary part of the frequencies increases with increasing mode number showing that the radiation is damping which in turn shows that the black hole is stable against massless Dirac perturbations. Real values of the frequencies increases in this case contradictory to the normal  $3 + 1$  black holes. Cho<sup>4</sup> obtained the Dirac quasinormal modes of Schwarzschild black hole using WKB approximation method. They

showed that the real part of quasinormal modes increases with angular momentum quantum number  $k$  and for a particular  $k$ , the real part of quasinormal modes decreases with increasing  $n$ . Jiliang<sup>6</sup> got the quasinormal modes of Schwarzschild black hole using continued fraction and Hill-determinant methods and obtained an increasing real part. In both cases the imaginary part increases with mode number showing damping and thus black holes in  $3 + 1$  dimensions are stable against Dirac perturbation. This result agrees with the present work for  $2 + 1$  black hole.

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### References

1. K. D. Kokkotas and B. G. Schmidt, Living Rev. Relativ. **2** 2 (1999)
2. C. V. Vishveshwara, Nature (London) **227**, 936 (1970)
3. S. Chandrasekhar and S. Detweiler, Proc. R. Soc. A **344** 441 (1975)
4. H. T. Cho, Phys. Rev. D **68** 024003 (2003)
5. A. Zhidenko, Class. Quant. Grav. **21**, 273 (2004)
6. Jiliang Jing, Phys. Rev. D **69**, 084009 (2004)
7. V. P. Frolov and I. D. Novikov, Black Hole Physics: Basic Concept and New Developments (Kluwer Academic, Dordrecht, 1998)
8. B. F. Schutz and C. M. Will, Astrophys. J. Lett. Ed. **291** L33 (1985)
9. S. Iyer and C. M. Will, Phys. Rev. D **35**, 3621 (1987)
10. S. Iyer, Phys. Rev. D **35**, 3632 (1987)
11. M. Banados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992)
12. V. Cardoso and J. P. S. Lemos Phys. Rev. D **63** 124015 (2001)
13. S. Fernando, Gen. Rel. Gravitation **36** 71-82 (2004)
14. G. Mandal et.al Mod. Phys. Lett. A **6** 1685 (1991)
15. S. Sebastian and V. C. Kuriakose Mod. Phys. Lett. A **28** 1350149 (2013)
16. K. C. K. Chan, R. B. Mann, Phys. Rev. D **50** 6385 (1994)
17. D. R. Brill and J. A. Wheeler, Rev. Mod. Phys. **29**, 465 (1957)